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New examples of indefinite hyper-Kähler symmetric spaces

Ines Kath[a], Martin Olbrich[b,*]

[a] Max-Planck-Institut für Mathematik in den Naturwissenschaften, Inselstraße 22-26, D-04103 Leipzig, Germany
[b] University of Luxembourg, Mathematics Laboratory, 162 A, Avenue de la Faïencerie, L-1511 Luxembourg, Luxembourg

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Abstract

Following the approach to pseudo-Riemannian symmetric spaces developed in [I. Kath, M. Olbrich, On the structure of pseudo-Riemannian symmetric spaces, arXiv:math.DG/0408249, 2004] we exhibit examples of indefinite hyper-Kähler symmetric spaces with non-abelian holonomy. Moreover, we classify indecomposable hyper-Kähler symmetric spaces whose metric has signature $(4, 4n)$. Such spaces exist if and only if $n \in \{0, 1, 3\}$.

Keywords: Pseudo-Riemannian symmetric spaces; Special geometries; Hyper-Kähler geometry; Metric Lie algebras

1. Introduction

The theory of special pseudo-Riemannian geometries has been steadily developing for some years now. Hyper-Kähler geometry is certainly one of the most important of these geometries. Recall that a (pseudo-Riemannian) hyper-Kähler manifold is a tuple (M, g, I, J), where (M, g) is a pseudo-Riemannian manifold, and I, J are two parallel anticommuting almost complex structures on M which preserve the scalar product g. In particular, $\dim M$ and the index of g are divisible by 4. It is natural to look first for symmetric examples of such manifolds, i.e. hyper-Kähler symmetric spaces, and to try to classify them. It is well-known that there are no non-flat Riemannian hyper-Kähler symmetric spaces. However, the pseudo-Riemannian situation is quite different. It turns out that there is a large amount of indefinite hyper-Kähler symmetric spaces including many continuous families. It is a natural task to describe all these spaces and possibly to classify them. However, this seems to be rather difficult. A first approach to a classification was given by Alekseevsky and Cortés [2]. They found a nice description of the isometry classes of hyper-Kähler symmetric spaces by quartic polynomials satisfying certain equations. Moreover, they exhibited a series of special solutions, which we want to call tame here. All hyper-Kähler symmetric spaces that correspond to these tame solutions have neutral signature and an abelian holonomy group of very special structure. However, it remains unclear how to find *all* solutions of the relevant equations, compare the discussion below. Further detail on the Alekseevsky–Cortés approach will be given in Section 5, Remark 5.3.

In [9] we developed a systematic approach to the construction and classification of pseudo-Riemannian symmetric spaces. It is based on the fact that the Lie algebra of the transvection group of a symmetric space can be obtained

* Corresponding author.
E-mail addresses: ikath@mis.mpg.de (I. Kath), martin.olbrich@uni.lu (M. Olbrich).

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from simpler objects by a canonical extension procedure, called quadratic extension. The above mentioned paper [2] by Alekseevsky and Cortés motivated us to test our method in the hyper-Kähler case. It is the aim of the present paper to shed some new light to the theory of hyper-Kähler symmetric spaces from this rather different perspective.

Simply connected hyper-Kähler symmetric spaces can be described by so-called hyper-Kähler symmetric triples. Such a triple consists of the Lie algebra of the transvection group of the symmetric space, an invariant non-degenerate inner product on this Lie algebra and a certain $Sp(1)$-action which is called quaternionic grading, see Section 2 for an exact definition. In Section 4 we will describe an extension procedure which yields a hyper-Kähler symmetric triple starting with a Lie algebra with quaternionic grading $(\mathfrak{l}, \Phi_{\mathfrak{l}})$, a pseudo-Euclidean vector space $\mathfrak{a}$ with quaternionic grading $\Phi_{\mathfrak{a}}$ and a pair $(\alpha, \gamma) \in \mathrm{Hom}(\bigwedge^2 \mathfrak{l}, \mathfrak{a}) \oplus \mathrm{Hom}(\bigwedge^3 \mathfrak{l}, \mathbb{R})$ of $Sp(1)$-invariant forms satisfying certain cocycle conditions. Note that the transvection group of a hyper-Kähler symmetric space is always nilpotent, see Corollary 2.4. Therefore the same is true for the Lie algebra $\mathfrak{l}$.

In Section 5 we will use this general construction to give concrete examples. We construct hyper-Kähler symmetric triples whose associated symmetric spaces have a metric of signature $(4n+4, 4n+12)$, $n \geq 0$, and a non-abelian holonomy group. Using the tangent bundle construction we obtain further examples which also have non-abelian holonomy. On the other hand, Alekseevsky and Cortés [2] had stated a classification of hyper-Kähler symmetric spaces, which would imply that any hyper-Kähler symmetric space has abelian holonomy. Indeed, the classification in [2] relied on the statement that all solutions of the considered equations are tame. However, our examples show that this statement is false and that there are many more hyper-Kähler symmetric spaces than claimed in [2]. This has also consequences for the results in [1,6]. For more information see Remark 5.3. After we had discovered the first counterexample to the classification in [2] (Example 1 in this paper) Cortés reconsidered the situation in [5]. He admitted the mistake and showed that the Alekseevsky–Cortés approach yields a classification if one considers only hyper-Kähler symmetric spaces with abelian holonomy. More exactly, he proved that all hyper-Kähler symmetric spaces with abelian holonomy correspond to tame solutions. For additional information see Remark 5.3.

Furthermore, in Section 6 we show that there is a canonical way to represent each hyper-Kähler symmetric triple as an extension of the kind described above. Such extensions of $(\mathfrak{l}, \Phi_{\mathfrak{l}})$ by $\mathfrak{a}$ which are associated with a hyper-Kähler symmetric space in this canonical way are classified by a cohomology set $\mathcal{H}^2_Q(\mathfrak{l}, \Phi_{\mathfrak{l}}, \mathfrak{a})_\sharp$ (introduced in Section 3). The isomorphism classes of underlying hyper-Kähler symmetric triples are in correspondence with the orbits of the action of the product of the automorphism groups of $(\mathfrak{l}, \Phi_{\mathfrak{l}})$ and $\mathfrak{a}$ on $\mathcal{H}^2_Q(\mathfrak{l}, \Phi_{\mathfrak{l}}, \mathfrak{a})_\sharp$. This yields a general classification scheme for hyper-Kähler symmetric triples. Theorem 6.1 states the corresponding result for indecomposable triples, i.e. for the objects one really wants to classify. We remark that the comparison of our extension method with the classification of hyper-Kähler symmetric spaces with abelian holonomy in [5] yields the following. If a hyper-Kähler symmetric triple is canonically represented as an extension of $(\mathfrak{l}, \Phi_{\mathfrak{l}})$ by $\mathfrak{a}$ then the holonomy of an associated hyper-Kähler symmetric space is abelian if and only if $\mathfrak{l}$ is abelian.

The general classification scheme described above can be used to find explicit classification results (i.e. lists) if one considers only hyper-Kähler symmetric spaces with a metric of a given small index. In Section 7 we will demonstrate this for indecomposable spaces of index 4. They are exhausted by the flat space $\mathbb{H}$, a one-parameter family of hyper-Kähler symmetric spaces of signature (4, 4) having abelian holonomy, and one single space of signature (4, 12) with non-abelian holonomy (the one constructed in Section 5, Example 1). See Theorem 7.4 for the precise statement.

2. Hyper-Kähler symmetric triples

We will say that $(\mathfrak{l}, \Phi_{\mathfrak{l}})$ is a *Lie algebra with quaternionic grading*, if $\mathfrak{l}$ is a Lie algebra and $\Phi_{\mathfrak{l}} : Sp(1) \to \mathrm{Aut}(\mathfrak{l})$ is an $Sp(1)$-action of the following kind. We assume that $\mathfrak{l} = \mathfrak{l}_+ \oplus \mathfrak{l}_-$ such that the representation of $Sp(1)$ on $\mathfrak{l}_+$ is trivial and the representation of $Sp(1)$ on $\mathfrak{l}_-$ is a multiple of the standard representation. In particular, $\mathfrak{l}_-$ is a left $\mathbb{H}$-module. If in addition $[\mathfrak{l}_-, \mathfrak{l}_-] = \mathfrak{l}_+$ we call the grading *proper*.

Similarly, we will say that $(\mathfrak{a}, \langle\cdot,\cdot\rangle_{\mathfrak{a}}, \Phi_{\mathfrak{a}})$ (or $\mathfrak{a}$ in abbreviated notation) is a *vector space with orthogonal quaternionic grading*, if $(\mathfrak{a}, \Phi_{\mathfrak{a}})$ is an abelian Lie algebra with quaternionic grading and the image of $\Phi_{\mathfrak{a}}$ is in $O(\mathfrak{a}, \langle\cdot,\cdot\rangle_{\mathfrak{a}})$.

Proposition 2.1. *If* $(\mathfrak{l}, \Phi_{\mathfrak{l}})$ *is a Lie algebra with proper quaternionic grading, then* $\mathfrak{l}$ *is nilpotent.*

Proof. We consider the semi-simple Lie algebra $\mathfrak{s} = \mathfrak{l}/\mathfrak{r}$, where $\mathfrak{r}$ is the solvable radical of $\mathfrak{l}$. The grading $\Phi_{\mathfrak{l}}$ induces a proper quaternionic grading $\Phi_{\mathfrak{s}}$. Being a connected subgroup of the automorphism group of $\mathfrak{s}$, the image of $\Phi_{\mathfrak{s}}$

consists of *inner* automorphisms which respect the decomposition $\mathfrak{s} = \mathfrak{s}_+ \oplus \mathfrak{s}_-$. Therefore its Lie algebra can be identified with a subalgebra $\mathfrak{k} \subset \mathfrak{s}_+$. It follows that the adjoint representation of $\mathfrak{k}$ on $\mathfrak{s}_+$, and hence on $\mathfrak{k}$, is trivial. On the other hand, $\mathfrak{k} \cong \mathfrak{sp}(1)$ unless $\mathfrak{s}_- = \{0\}$. We conclude that $\mathfrak{s} = \{0\}$, i.e., $\mathfrak{l}$ is solvable.

We finish the proof by showing that solvable Lie algebras $\mathfrak{l} = \mathfrak{l}_+ \oplus \mathfrak{l}_-$, $\mathfrak{l}_+ = [\mathfrak{l}_-, \mathfrak{l}_-]$, admitting an automorphism F such that $F^2\big|_{\mathfrak{l}_-} = -\mathrm{Id}$ and $F|_{\mathfrak{l}_+} = \mathrm{Id}$ are nilpotent. Indeed, $F = \Phi_{\mathfrak{l}}(i)$ is such an automorphism.

We look at the decreasing chain of ideals of $\mathfrak{l}$

$$\mathfrak{l} := R_0(\mathfrak{l}) \supset R_1(\mathfrak{l}) \supset R_2(\mathfrak{l}) \ldots,$$

where $R_{k+1}(\mathfrak{l})$ is the minimal $\mathfrak{l}$-ideal in $R_k(\mathfrak{l})$ such that the induced action of $\mathfrak{l}$ on $R_k(\mathfrak{l})/R_{k+1}(\mathfrak{l})$ is semi-simple. Note that these ideals are F-invariant. In fact, they are under all automorphisms of $\mathfrak{l}$. There exists a number m such that $R_m(\mathfrak{l}) = \{0\}$.

We look at the semi-simple representation of $\mathfrak{l}$ on the complexification W_k of $R_k(\mathfrak{l})/R_{k+1}(\mathfrak{l})$. Since $\mathfrak{l}'$ is a nilpotent ideal of $\mathfrak{l}$, the Lie algebra $\mathfrak{l}_+ = [\mathfrak{l}_-, \mathfrak{l}_-] \subset \mathfrak{l}'$ acts trivially on W_k. It follows that W_k is the direct sum of weight spaces E_λ with $\lambda \in (\mathfrak{l}_-^*)_{\mathbb{C}}$. Note that F acts naturally on $(\mathfrak{l}_-^*)_{\mathbb{C}}$ and on W_k with the property $F(E_\lambda) = E_{F(\lambda)}$. Assume that $\lambda \neq 0$. Then the elements $F^n(\lambda)$, $n = 0, 1, 2, 3$, are pairwise different. Therefore the sum of the weight spaces $E_{F^n(\lambda)}$, $n = 0, 1, 2, 3$, is direct. Take $v \in E_\lambda$. Then $F^n(v) \in E_{F^n(\lambda)}$, and

$$v^- := v - F(v) + F^2(v) - F^3(v)$$

satisfies $F(v^-) = -v^-$. However, the only possible eigenvalues of F on W_k are 1, i and $-i$. We conclude that $v^- = 0$, hence $v = 0$. It follows that $W_k = E_0$, i.e. $\mathfrak{l}$ acts trivially on W_k. We conclude that $\mathfrak{l}$ is nilpotent. □

Definition 2.2. A hyper-Kähler symmetric triple is a triple $(\mathfrak{g}, \Phi_{\mathfrak{g}}, \langle\cdot\,,\cdot\rangle_{\mathfrak{g}})$, where $(\mathfrak{g}, \Phi_{\mathfrak{g}})$ is a Lie algebra with proper quaternionic grading and $\langle\cdot\,,\cdot\rangle_{\mathfrak{g}}$ is an $\mathfrak{l}-$ and $\Phi_{\mathfrak{l}}$-invariant non-degenerate symmetric bilinear symmetric bilinear form.

There is an obvious notion of isomorphism between hyper-Kähler symmetric triples. Note that $\theta_{\mathfrak{g}} := \Phi_{\mathfrak{g}}(-1)$ is an isometric involution of $\mathfrak{g}$ and $(\mathfrak{g}, \theta_{\mathfrak{g}}, \langle\cdot\,,\cdot\rangle_{\mathfrak{g}})$ is a symmetric triple in the sense of [9].

Proposition 2.3. *The Lie algebra of the transvection group of a hyper-Kähler symmetric space carries the structure of a hyper-Kähler symmetric triple in a canonical way. There is a one-to-one correspondence between isometry classes of simply connected hyper-Kähler symmetric spaces and isomorphism classes of hyper-Kähler symmetric triples.*

Proof. The proposition is a slight variant of the well-known correspondence between pseudo-Riemannian symmetric spaces and symmetric triples (see [4], compare [9], Section 2). If $(\mathfrak{g}, \Phi_{\mathfrak{g}}, \langle\cdot\,,\cdot\rangle_{\mathfrak{g}})$ is a hyper-Kähler symmetric triple and (M, g) is a symmetric space with associated symmetric triple $(\mathfrak{g}, \theta_{\mathfrak{g}}, \langle\cdot\,,\cdot\rangle_{\mathfrak{g}})$, then $I = \Phi_{\mathfrak{g}}(i)\big|_{\mathfrak{g}_-}$ and $J = \Phi_{\mathfrak{g}}(j)\big|_{\mathfrak{g}_-}$ are $\mathfrak{g}_+$-invariant anticommuting complex structures on $\mathfrak{g}_- \cong T_oM$ respecting the metric and therefore induce a hyper-Kähler structure on M.

The only non-obvious point is the opposite direction. Let (M, g, I, J) be a hyper-Kähler symmetric space and let $(\mathfrak{g}, \theta_{\mathfrak{g}}, \langle\cdot\,,\cdot\rangle_{\mathfrak{g}})$ be the associated symmetric triple. Then $I, J, K := IJ$ span a Lie algebra $\mathfrak{k} \cong \mathfrak{sp}(1)$ which acts orthogonally on $\mathfrak{g}_- \cong T_oM$. This action commutes with the one of $\mathfrak{g}_+$. We extend the $\mathfrak{k}$-action to $\mathfrak{g}$ by the trivial action on $\mathfrak{g}_+$. For $X, Y \in \mathfrak{g}_-$, $Z \in \mathfrak{g}_+$, and $Q \in \mathfrak{k}$ we compute

$$\begin{aligned}\langle [QX, Y] + [X, QY], Z\rangle &= \langle QX, [Y, Z]\rangle + \langle X, [QY, Z]\rangle \\ &= \langle QX, [Y, Z]\rangle + \langle X, Q[Y, Z]\rangle = 0.\end{aligned}$$

It follows that $\mathfrak{k}$ acts by derivations on $\mathfrak{g}$. Integrating the resulting homomorphism of $\mathfrak{sp}(1)$ into the derivations of $\mathfrak{g}$ we obtain the desired homomorphism $\Phi_{\mathfrak{g}} : Sp(1) \to \mathrm{Aut}(\mathfrak{g})$. □

Taking Proposition 2.1 into account we obtain

Corollary 2.4. *The transvection group of a hyper-Kähler symmetric space is nilpotent.*

We remark that solvability of the transvection group was already shown in [2].

A hyper-Kähler symmetric triple is called decomposable, if it is the orthogonal direct sum of two non-zero hyper-Kähler symmetric triples, and indecomposable otherwise. The simply connected hyper-Kähler symmetric spaces which correspond to indecomposable hyper-Kähler triples are precisely those which are indecomposable in the differential geometric sense. This follows from the de Rham–Wu decomposition theorem.

The underlying metric Lie algebra of a hyper-Kähler symmetric triple $(\mathfrak{g}, \Phi_{\mathfrak{g}}, \langle\cdot,\cdot\rangle_{\mathfrak{g}})$ is the tuple $(\mathfrak{g}, \langle\cdot,\cdot\rangle_{\mathfrak{g}})$. There are analogous notions of indecomposability for symmetric triples and for metric Lie algebras.

Lemma 2.5. *Let $(\mathfrak{g}, \Phi_{\mathfrak{g}}, \langle\cdot,\cdot\rangle_{\mathfrak{g}})$ be a hyper-Kähler symmetric triple with $\mathfrak{g}$ non-abelian. Then the following conditions are equivalent:*

(i) *$(\mathfrak{g}, \Phi_{\mathfrak{g}}, \langle\cdot,\cdot\rangle_{\mathfrak{g}})$ is indecomposable.*
(ii) *The symmetric triple $(\mathfrak{g}, \theta_{\mathfrak{g}}, \langle\cdot,\cdot\rangle_{\mathfrak{g}})$ is indecomposable.*
(iii) *The metric Lie algebra $(\mathfrak{g}, \langle\cdot,\cdot\rangle_{\mathfrak{g}})$ is indecomposable.*

Proof. The implications (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) are obvious. We have to prove (i) $\Rightarrow$ (iii).

Assume (i). We first observe that the center $\mathfrak{z}(\mathfrak{g})$ is isotropic, i.e. $\mathfrak{z}(\mathfrak{g}) \subset \mathfrak{z}(\mathfrak{g})^{\perp} = \mathfrak{g}'$. Indeed, any $Sp(1)$-invariant complement of $\mathfrak{z}(\mathfrak{g}) \cap \mathfrak{z}(\mathfrak{g})^{\perp}$ in $\mathfrak{z}(\mathfrak{g})$ is a non-degenerate abelian ideal of $\mathfrak{g}$. It has to be zero by assumption. Let

$$\mathfrak{g} = \bigoplus_{k=1}^{n} \mathfrak{i}_k \tag{1}$$

be an orthogonal decomposition of $(\mathfrak{g}, \langle\cdot,\cdot\rangle_{\mathfrak{g}})$ into non-trivial indecomposable ideals. Let $\mathfrak{j} \subset \mathfrak{g}$ be a further non-degenerate indecomposable ideal. According to [3], Theorem 3, there exists $l \in \{1, \ldots, n\}$ such that $\mathfrak{j}' = \mathfrak{i}_l{}'$. In particular, the set of derivatives of non-degenerate indecomposable ideals is finite. Therefore the action of the connected group $Sp(1)$ on this set is trivial. It follows that the ideals $\mathfrak{i}_1{}'$ and $(\bigoplus_{k=2}^{n} \mathfrak{i}_k')^{\perp} = \mathfrak{i}_1 + \mathfrak{z}(\mathfrak{g})$ are $Sp(1)$-invariant. The above observation concerning the center yields $\mathfrak{z}(\mathfrak{i}_1) = \mathfrak{i}_1{}' \cap \mathfrak{z}(\mathfrak{g})$. It follows that this ideal is $Sp(1)$-invariant, too. Therefore $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$ induces an $Sp(1)$-equivariant surjective map

$$\psi : \mathfrak{i}_1 + \mathfrak{z}(\mathfrak{g}) \longrightarrow \mathfrak{z}(\mathfrak{i}_1)^*.$$

Note that $\mathfrak{i}_1{}' \subset \ker(\psi)$. Let $s : \mathfrak{z}(\mathfrak{i}_1)^* \to \mathfrak{i}_1 + \mathfrak{z}(\mathfrak{g})$ be an $Sp(1)$-equivariant section of ψ. Then $\widetilde{\mathfrak{i}_1} := \mathfrak{i}_1{}' \oplus s(\mathfrak{z}(\mathfrak{i}_1)^*)$ is a non-degenerate and $Sp(1)$-invariant ideal of $\mathfrak{g}$. Moreover, the projection $\widetilde{\mathfrak{i}_1} \to \mathfrak{i}_1$ with respect to (1) is an isomorphism. Indecomposability of the triple $(\mathfrak{g}, \Phi_{\mathfrak{g}}, \langle\cdot,\cdot\rangle_{\mathfrak{g}})$ now implies $\mathfrak{g} = \widetilde{\mathfrak{i}_1}$. Thus $\mathfrak{g} \cong \mathfrak{i}_1$ is indecomposable as a metric Lie algebra. □

3. Quaternionic quadratic cohomology

Let us first recall the notion of quadratic cohomology introduced in [8]. Let $\mathfrak{l}$ be a finite-dimensional Lie algebra. An *orthogonal $\mathfrak{l}$-module* is a tuple $(\rho, \mathfrak{a}, \langle\cdot,\cdot\rangle_{\mathfrak{a}})$ (also $\mathfrak{a}$ in abbreviated notation) such that ρ is a representation of the Lie algebra $\mathfrak{l}$ on the finite-dimensional real vector space $\mathfrak{a}$ and $\langle\cdot,\cdot\rangle_{\mathfrak{a}}$ is a scalar product on $\mathfrak{a}$ such that $\rho(L) \in \mathfrak{so}(\mathfrak{a}, \langle\cdot,\cdot\rangle_{\mathfrak{a}})$ for all $L \in \mathfrak{l}$.

For $\mathfrak{l}$ and (any $\mathfrak{l}$-module) $\mathfrak{a}$ we have the standard cochain complex $(C^*(\mathfrak{l}, \mathfrak{a}), d)$, where $C^p(\mathfrak{l}, \mathfrak{a}) = \mathrm{Hom}\,(\bigwedge^p \mathfrak{l}, \mathfrak{a})$ and we have the corresponding cocycle groups $Z^p(\mathfrak{l}, \mathfrak{a})$ and cohomology groups $H^p(\mathfrak{l}, \mathfrak{a})$. If $\mathfrak{a}$ is the one-dimensional trivial representation, then we denote this cochain complex also by $C^*(\mathfrak{l})$.

We have a product

$$\langle\cdot \wedge \cdot\rangle : C^p(\mathfrak{l}, \mathfrak{a}) \times C^q(\mathfrak{l}, \mathfrak{a}) \longrightarrow C^{p+q}(\mathfrak{l})$$

defined by the composition

$$C^p(\mathfrak{l}, \mathfrak{a}) \times C^q(\mathfrak{l}, \mathfrak{a}) \xrightarrow{\wedge} C^{p+q}(\mathfrak{l}, \mathfrak{a} \otimes \mathfrak{a}) \xrightarrow{\langle\cdot,\cdot\rangle_{\mathfrak{a}}} C^p(\mathfrak{l}).$$

The group of quadratic 1-cochains is the group

$$\mathcal{C}^1_Q(\mathfrak{l}, \mathfrak{a}) = C^1(\mathfrak{l}, \mathfrak{a}) \oplus C^2(\mathfrak{l})$$

with group operation defined by

$$(\tau_1, \sigma_1) * (\tau_2, \sigma_2) = \left(\tau_1 + \tau_2, \sigma_1 + \sigma_2 + \frac{1}{2}\langle\tau_1 \wedge \tau_2\rangle\right).$$

We consider now the set

$$\mathcal{Z}^2_Q(\mathfrak{l},\mathfrak{a}) = \left\{(\alpha,\gamma) \in C^2(\mathfrak{l},\mathfrak{a}) \oplus C^3(\mathfrak{l}) \mid \mathrm{d}\alpha = 0,\ \mathrm{d}\gamma = \frac{1}{2}\langle \alpha \wedge \alpha\rangle\right\}$$

whose elements are called quadratic 2-cocycles. The group $\mathcal{C}^1_Q(\mathfrak{l},\mathfrak{a})$ acts on $\mathcal{Z}^2_Q(\mathfrak{l},\mathfrak{a})$ by

$$(\alpha,\gamma)(\tau,\sigma) = \left(\alpha + \mathrm{d}\tau, \gamma + \mathrm{d}\sigma + \left\langle \left(\alpha + \frac{1}{2}\mathrm{d}\tau\right) \wedge \tau \right\rangle\right).$$

Ordinary quadratic cohomology is then the orbit space of this action:

$$\mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a}) = \mathcal{Z}^2_Q(\mathfrak{l},\mathfrak{a})/\mathcal{C}^1_Q(\mathfrak{l},\mathfrak{a}).$$

Now let $(\mathfrak{l},\Phi_\mathfrak{l})$ be a Lie algebra with quaternionic grading and let $(\mathfrak{a},\Phi_\mathfrak{a})$ be a vector space with orthogonal quaternionic grading. We consider $\mathfrak{a}$ as a trivial $\mathfrak{l}$-module. Then $\Phi_\mathfrak{a}$ and $\Phi_\mathfrak{l}$ define $Sp(1)$-actions on $\mathcal{Z}^2_Q(\mathfrak{l},\mathfrak{a})$ and $\mathcal{C}^1_Q(\mathfrak{l},\mathfrak{a})$. More precisely, for $q \in Sp(1)$ the pair of morphisms $(\Phi_\mathfrak{l}(q),\Phi_\mathfrak{a}(q)^{-1})$ induces pullback maps on $C^2(\mathfrak{l},\mathfrak{a}) \oplus C^3(\mathfrak{l})$ and on $\mathcal{C}^1_Q(\mathfrak{l},\mathfrak{a}) = C^1(\mathfrak{l},\mathfrak{a}) \oplus C^2(\mathfrak{l})$, which leave invariant $\mathcal{Z}^2_Q(\mathfrak{l},\mathfrak{a})$ and are compatible with the $\mathcal{C}^1_Q(\mathfrak{l},\mathfrak{a})$-action on $\mathcal{Z}^2_Q(\mathfrak{l},\mathfrak{a})$. We consider the sets of invariants

$$\mathcal{Z}^2_Q(\mathfrak{l},\Phi_\mathfrak{l},\mathfrak{a}) := \mathcal{Z}^2_Q(\mathfrak{l},\mathfrak{a})^{Sp(1)} \quad \text{and} \quad \mathcal{C}^1_Q(\mathfrak{l},\Phi_\mathfrak{l},\mathfrak{a}) := \mathcal{C}^1_Q(\mathfrak{l},\mathfrak{a})^{Sp(1)}.$$

The group $\mathcal{C}^1_Q(\mathfrak{l},\Phi_\mathfrak{l},\mathfrak{a})$ acts on $\mathcal{Z}^2_Q(\mathfrak{l},\Phi_\mathfrak{l},\mathfrak{a})$ and we can define the (second) quaternionic quadratic cohomology by

$$\mathcal{H}^2_Q(\mathfrak{l},\Phi_\mathfrak{l},\mathfrak{a}) := \mathcal{Z}^2_Q(\mathfrak{l},\Phi_\mathfrak{l},\mathfrak{a})/\mathcal{C}^1_Q(\mathfrak{l},\Phi_\mathfrak{l},\mathfrak{a}).$$

For a Lie algebra $\mathfrak{l}$ we denote by $\mathfrak{l}^1 = \mathfrak{l},\ldots,\mathfrak{l}^k = [\mathfrak{l},\mathfrak{l}^{k-1}],\ldots$ the lower central series.

Definition 3.1. Let $(\mathfrak{l},\Phi_\mathfrak{l})$ be a Lie algebra with a proper quaternionic grading and let $\mathfrak{a}$ be a vector space with orthogonal quaternionic grading. Let m be such that $\mathfrak{l}^{m+2} = 0$. Let $(\alpha,\gamma) \in \mathcal{Z}^2_Q(\mathfrak{l},\Phi_\mathfrak{l},\mathfrak{a})$. Then the cohomology class $[\alpha,\gamma] \in \mathcal{H}^2_Q(\mathfrak{l},\Phi_\mathfrak{l},\mathfrak{a})$ is called admissible if and only if the following conditions (T), (A_k) and (B_k) hold for all $0 \le k \le m$.

- (T) $\mathfrak{a}_+ = \alpha(\ker[\cdot\,,\cdot]_{\mathfrak{l}_-})$.
- (A_k) Let $L_0 \in \mathfrak{z}(\mathfrak{l}) \cap \mathfrak{l}^{k+1}$ be such that there exist elements $A_0 \in \mathfrak{a}$ and $Z_0 \in (\mathfrak{l}^{k+1})^*$ satisfying
  - (i) $\alpha(L,L_0) = 0$,
  - (ii) $\gamma(L,L_0,\cdot) = -\langle A_0,\alpha(L,\cdot)\rangle_\mathfrak{a} + \langle Z_0,[L,\cdot]_\mathfrak{l}\rangle$ as an element of $(\mathfrak{l}^{k+1})^*$,

  for all $L \in \mathfrak{l}$, then $L_0 = 0$.
- (B_k) The subspace $\alpha(\ker[\cdot\,,\cdot]_{\mathfrak{l}\otimes\mathfrak{l}^{k+1}}) \subset \mathfrak{a}$ is non-degenerate, where $\ker[\cdot\,,\cdot]_{\mathfrak{l}\otimes\mathfrak{l}^{k+1}}$ is the kernel of the map $[\cdot\,,\cdot] : \mathfrak{l}\otimes\mathfrak{l}^{k+1} \to \mathfrak{l}$.

We denote the set of all admissible cohomology classes by $\mathcal{H}^2_Q(\mathfrak{l},\Phi_\mathfrak{l},\mathfrak{a})_\sharp$.

The admissibility conditions are specializations of the ones in [9], Definition 5.2, to the case that $\mathfrak{l}$ is nilpotent and that the representation of $\mathfrak{l}$ on $\mathfrak{a}$ is trivial. As in [9], they do not depend on the choice of the cocycle representing the cohomology class $[\alpha,\gamma]$.

Now let $\mathfrak{l}_i$, $\mathfrak{a}_i$, $i = 1,2$, be Lie algebras (vector spaces, resp.) with (orthogonal) quaternionic grading. We form $\mathfrak{l} = \mathfrak{l}_1 \oplus \mathfrak{l}_2$, $\mathfrak{a} = \mathfrak{a}_1 \oplus \mathfrak{a}_2$ and consider the corresponding projections $\mathrm{pr}_i : \mathfrak{l} \to \mathfrak{l}_i$. If $(\alpha_i,\gamma_i) \in \mathcal{Z}^2_Q(\mathfrak{l}_i,\Phi_{\mathfrak{l}_i},\mathfrak{a}_i)$, then

$$(\mathrm{pr}_1^*\alpha_1 \oplus \mathrm{pr}_2^*\alpha_2, \mathrm{pr}_1^*\gamma_1 + \mathrm{pr}_2^*\gamma_2) \in \mathcal{Z}^2_Q(\mathfrak{l},\Phi_\mathfrak{l},\mathfrak{a}).$$

This operation induces a map

$$\oplus : \mathcal{H}^2_Q(\mathfrak{l}_1,\Phi_{\mathfrak{l}_1},\mathfrak{a}_1) \oplus \mathcal{H}^2_Q(\mathfrak{l}_2,\Phi_{\mathfrak{l}_2},\mathfrak{a}_2) \longrightarrow \mathcal{H}^2_Q(\mathfrak{l},\Phi_\mathfrak{l},\mathfrak{a}).$$

Definition 3.2. A cohomology class $\phi \in \mathcal{H}^2_Q(\mathfrak{l}, \Phi_{\mathfrak{l}}, \mathfrak{a})$ is called decomposable, if there are a decomposition $\mathfrak{l} = \mathfrak{l}_1 \oplus \mathfrak{l}_2$ into $\Phi_{\mathfrak{l}}$-invariant ideals, a $\Phi_{\mathfrak{a}}$-invariant orthogonal decomposition $\mathfrak{a} = \mathfrak{a}_1 \oplus \mathfrak{a}_2$ (at least one of these decompositions should be non-trivial), and cohomology classes $\phi_i \in \mathcal{H}^2_Q(\mathfrak{l}_i, \Phi_{\mathfrak{l}_i}, \mathfrak{a}_i)$ such that $\phi = \phi_1 \oplus \phi_2$. Otherwise we call ϕ indecomposable.

We denote the set of all indecomposable elements in $\mathcal{H}^2_Q(\mathfrak{l}, \Phi_{\mathfrak{l}}, \mathfrak{a})_\sharp$ by $\mathcal{H}^2_Q(\mathfrak{l}, \Phi_{\mathfrak{l}}, \mathfrak{a})_0$.

4. A construction method

Now we will present a construction method which yields hyper-Kähler symmetric triples starting with a Lie algebra with proper quaternionic grading $(\mathfrak{l}, \Phi_{\mathfrak{l}})$, a vector space $\mathfrak{a}$ with orthogonal quaternionic grading and an admissible cocycle in $\mathcal{Z}^2_Q(\mathfrak{l}, \Phi_{\mathfrak{l}}, \mathfrak{a})$ (i.e. a cocycle which represents an admissible cohomology class). This construction is a special case of the construction method for symmetric triples presented in [9], Section 4.2. In Section 6 we will see that each hyper-Kähler symmetric triple arises by this construction in a canonical way.

Let $(\mathfrak{l}, \Phi_{\mathfrak{l}})$ be a Lie algebra with proper quaternionic grading and let $(\mathfrak{a}, \langle\cdot\,,\cdot\rangle_{\mathfrak{a}}, \Phi_{\mathfrak{a}})$ be a vector space with orthogonal quaternionic grading. We consider the vector space

$$\mathfrak{d} := \mathfrak{l}^* \oplus \mathfrak{a} \oplus \mathfrak{l}.$$

Now we choose an admissible cocycle $(\alpha, \gamma) \in \mathcal{Z}^2_Q(\mathfrak{l}, \Phi_{\mathfrak{l}}, \mathfrak{a})$ and define a Lie bracket $[\cdot\,,\cdot] : \mathfrak{d} \times \mathfrak{d} \to \mathfrak{d}$ by $[\mathfrak{l}^* \oplus \mathfrak{a}, \mathfrak{l}^* \oplus \mathfrak{a}] = 0$ and

$$[L_1, L_2] = \gamma(L_1, L_2, \cdot) + \alpha(L_1, L_2) + [L_1, L_2]_{\mathfrak{l}} \tag{2}$$
$$[L, A] = -\langle A, \alpha(L, \cdot)\rangle \tag{3}$$
$$[L, Z] = \mathrm{ad}^*(L)(Z) \tag{4}$$

for $Z \in \mathfrak{l}^*$, $A \in \mathfrak{a}$ and $L, L_1, L_2 \in \mathfrak{l}$. Moreover we define an inner product $\langle\cdot\,,\cdot\rangle$ and a quaternionic grading Φ on $\mathfrak{d}$ by

$$\langle Z_1 + A_1 + L_1, Z_2 + A_2 + L_2\rangle := \langle A_1, A_2\rangle_{\mathfrak{a}} + Z_1(L_2) + Z_2(L_1)$$
$$\Phi(Z + A + L) := \Phi^*_{\mathfrak{l}}(Z) + \Phi_{\mathfrak{a}}(A) + \Phi_{\mathfrak{l}}(L)$$

for $Z, Z_1, Z_2 \in \mathfrak{l}^*$, $A, A_1, A_2 \in \mathfrak{a}$ and $L, L_1, L_2 \in \mathfrak{l}$.

Proposition 4.1. *The tuple $\mathfrak{d}_{\alpha,\gamma}(\mathfrak{l}, \Phi_{\mathfrak{l}}, \mathfrak{a}) := (\mathfrak{d}, \Phi, \langle\cdot\,,\cdot\rangle)$ is a hyper-Kähler symmetric triple. It is indecomposable if and only if $[\alpha, \gamma] \in \mathcal{H}^2_Q(\mathfrak{l}, \Phi_{\mathfrak{l}}, \mathfrak{a})_\sharp$ is indecomposable.*

Proof. This follows from the results of [9], especially Proposition 4.1, Lemma 5.2, and Proposition 6.2. □

Remark 4.2. If $(\alpha, \gamma) \in \mathcal{Z}^2_Q(\mathfrak{l}, \Phi_{\mathfrak{l}}, \mathfrak{a})$ is arbitrary, then $\mathfrak{d}_{\alpha,\gamma}(\mathfrak{l}, \Phi_{\mathfrak{l}}, \mathfrak{a})$ is still a Lie algebra with quaternionic grading and an invariant inner product. One needs some conditions on the cocycle in order to ensure properness. For this purpose, conditions much weaker than admissibility would suffice (e.g. (T) together with (A_0)). However, we can detect indecomposability of $\mathfrak{d}_{\alpha,\gamma}(\mathfrak{l}, \Phi_{\mathfrak{l}}, \mathfrak{a})$ for admissible cocycles (α, γ) only. The main reason for considering the admissibility conditions is that any hyper-Kähler symmetric triple is isomorphic to $\mathfrak{d}_{\alpha,\gamma}(\mathfrak{l}, \Phi_{\mathfrak{l}}, \mathfrak{a})$ for an essentially unique tuple $(\mathfrak{l}, \Phi_{\mathfrak{l}}, \mathfrak{a}, [\alpha, \gamma] \in \mathcal{H}^2_Q(\mathfrak{l}, \Phi_{\mathfrak{l}}, \mathfrak{a})_\sharp)$, see Theorem 6.1 below.

Remark 4.3. The signature (p, q) of a hyper-Kähler symmetric triple $(\mathfrak{g}, \Phi_{\mathfrak{g}}, \langle\cdot\,,\cdot\rangle_{\mathfrak{g}})$ is defined to be the signature of the restriction of $\langle\cdot\,,\cdot\rangle_{\mathfrak{g}}$ to $\mathfrak{g}_-$. Here p is the dimension of a maximal negative definite subspace of $\mathfrak{g}_-$. Sometimes p is called the index of the triple. The signature (the index) of a hyper-Kähler symmetric triple equals the signature (the index) of the metric of any pseudo-Riemannian (hyper-Kähler) symmetric space which is associated with the triple. The signature (p, q) of the above constructed hyper-Kähler symmetric triple $\mathfrak{d}_{\alpha,\gamma}(\mathfrak{l}, \Phi_{\mathfrak{l}}, \mathfrak{a})$ is determined by the signature $(p_{\mathfrak{a}}, q_{\mathfrak{a}})$ of the restriction of $\langle\cdot\,,\cdot\rangle_{\mathfrak{a}}$ to $\mathfrak{a}_-$ and the dimension of $\mathfrak{l}_-$:

$$(p, q) = (p_{\mathfrak{a}} + \dim \mathfrak{l}_-, q_{\mathfrak{a}} + \dim \mathfrak{l}_-). \tag{5}$$

5. Examples of hyper-Kähler symmetric spaces with non-abelian holonomy

Now we will use the method described above to construct various hyper-Kähler symmetric triples whose associated symmetric spaces have non-abelian holonomy. In particular, these spaces were missing in a classification result for hyper-Kähler symmetric triples claimed by Alekseevsky and Cortés [2].

Example 1

First let us define the structure of a Lie algebra with proper quaternionic grading on the vector space $\mathfrak{l}_0 = \mathbb{H} \oplus \operatorname{Im}\mathbb{H}$. For all $q \in \mathbb{H}$ we denote $(q, 0)$ only by q and $(0, i)$, $(0, j)$, $(0, k)$ by I, J, and K respectively. On $\mathfrak{l}_0$ we define a Lie bracket $[\cdot\,,\cdot]$ by

$$I, J, K \in \mathfrak{z}(\mathfrak{l}_0), \qquad [q_1, q_2] = (0, \operatorname{Im}\bar{q}_1 q_2) \in \mathbb{H} \oplus \operatorname{Im}\mathbb{H}.$$

Moreover, on $\mathfrak{l}_0 = \mathbb{H} \oplus \operatorname{Im}\mathbb{H}$ we consider the $Sp(1)$-action $\Phi_{\mathfrak{l}_0}$ which is the left multiplication on the first summand and which is trivial on the second summand.

Now we will describe a suitable vector space $\mathfrak{a}_0$ with orthogonal quaternionic grading such that $\mathcal{H}^2_Q(\mathfrak{l}_0, \Phi_{\mathfrak{l}_0}, \mathfrak{a}_0)_0$ is not empty. Let e_1, e_2, e_3 be the standard basis of $\mathbb{R}^3$ and define $A_1 = e_1 - e_2$, $A_2 = e_2 - e_3$ and $A_3 = e_3 - e_1$. We consider the 2-dimensional vector space $\mathfrak{a}_{\mathbb{R}} = \operatorname{span}\{A_1, A_2, A_3\}$. Let $\langle\cdot\,,\cdot\rangle$ be the restriction of the (positive definite) standard scalar product on $\mathbb{R}^3$ to $\mathfrak{a}_{\mathbb{R}}$ and let $\langle\cdot\,,\cdot\rangle_{\mathbb{H}}$ be the standard scalar product on $\mathbb{H}$, i.e. $\langle p, q\rangle_{\mathbb{H}} = \operatorname{Re}\bar{p}q$. Then $\langle\cdot\,,\cdot\rangle_0 := \langle\cdot\,,\cdot\rangle_{\mathbb{H}} \otimes \langle\cdot\,,\cdot\rangle$ is a scalar product on $\mathfrak{a}_0 := \mathbb{H} \otimes_{\mathbb{R}} \mathfrak{a}_{\mathbb{R}}$. Furthermore, if we let $\Phi_{\mathfrak{a}_0}$ be the left multiplication on $\mathfrak{a}_0$ we obtain a vector space with orthogonal quaternionic grading $(\mathfrak{a}_0, \langle\cdot\,,\cdot\rangle_0, \Phi_{\mathfrak{a}_0})$.

We define $\alpha_0 \in C^2(\mathfrak{l}_0, \mathfrak{a}_0)$ by

$$\alpha_0(q, I) = qiA_1, \qquad \alpha_0(q, J) = qjA_2, \qquad \alpha_0(q, K) = qkA_3 \tag{6}$$

for all $q \in \mathbb{H}$ and by $\alpha_0((\mathfrak{l}_0)_+, (\mathfrak{l}_0)_+) = \alpha_0((\mathfrak{l}_0)_-, (\mathfrak{l}_0)_-) = 0$. Furthermore, we define $\gamma_0 \in C^3(\mathfrak{l}_0)$ by $\gamma_0(I, J, K) = 2$ and $\gamma_0((\mathfrak{l}_0)_-, \mathfrak{l}_0, \mathfrak{l}_0) = 0$.

Lemma 5.1. *We have* $(\alpha_0, \gamma_0) \in \mathcal{Z}^2_Q(\mathfrak{l}_0, \Phi_{\mathfrak{l}_0}, \mathfrak{a}_0)$. *Moreover,* (α_0, γ_0) *is admissible and indecomposable.*

Proof. Obviously, α_0 and γ_0 are $Sp(1)$-invariant. Let us compute

$$\begin{aligned} \mathrm{d}\alpha_0(1, i, j) &= -\alpha_0([1, i], j) + \alpha_0([1, j], i) - \alpha_0([i, j], 1) \\ &= -\alpha_0(I, j) + \alpha_0(J, i) + \alpha_0(K, 1) \\ &= jiA_1 - ijA_2 - kA_3 = 0. \end{aligned}$$

Because of the $Sp(1)$-invariance of α_0 this implies

$$\mathrm{d}\alpha_0(1, i, k) = \mathrm{d}\alpha_0(1, j, k) = \mathrm{d}\alpha_0(i, j, k) = 0.$$

Since all other components of $\mathrm{d}\alpha_0$ vanish obviously, we obtain $\mathrm{d}\alpha_0 = 0$.

Now we will check the condition $\mathrm{d}\gamma_0 = \frac{1}{2}\langle\alpha_0 \wedge \alpha_0\rangle$. Obviously we have

$$\mathrm{d}\gamma_0(\mathfrak{l}_0, (\mathfrak{l}_0)_+, (\mathfrak{l}_0)_+, (\mathfrak{l}_0)_+) = \langle\alpha_0 \wedge \alpha_0\rangle(\mathfrak{l}_0, (\mathfrak{l}_0)_+, (\mathfrak{l}_0)_+, (\mathfrak{l}_0)_+) = 0$$

and

$$\mathrm{d}\gamma_0(\mathfrak{l}_0, (\mathfrak{l}_0)_-, (\mathfrak{l}_0)_-, (\mathfrak{l}_0)_-) = \langle\alpha_0 \wedge \alpha_0\rangle(\mathfrak{l}_0, (\mathfrak{l}_0)_-, (\mathfrak{l}_0)_-, (\mathfrak{l}_0)_-) = 0.$$

By $Sp(1)$-invariance of (α_0, γ_0) it remains to prove

$$\mathrm{d}\gamma_0(1, q, P, Q) = \frac{1}{2}\langle\alpha_0 \wedge \alpha_0\rangle(1, q, P, Q)$$

for all imaginary $q \in \mathbb{H}$ and all $P, Q \in \{I, J, K\}$. We will show this for $q = i$. The remaining equations can be proved similarly. We have

$$\frac{1}{2}\langle\alpha_0 \wedge \alpha_0\rangle(1, i, I, J) = \langle\alpha_0(I, 1), \alpha_0(i, J)\rangle_0 + \langle\alpha_0(i, I), \alpha_0(1, J)\rangle_0$$

$$
\begin{aligned}
&= \langle -iA_1, kA_2\rangle_0 + \langle -A_1, jA_2\rangle_0 \\
&= 0 = -\gamma_0(I, I, J) = \mathrm{d}\gamma_0(1, i, I, J) \\
\frac{1}{2}\langle \alpha_0 \wedge \alpha_0\rangle(1, i, I, K) &= \langle \alpha_0(I, 1), \alpha_0(i, K)\rangle_0 + \langle \alpha_0(i, I), \alpha_0(1, K)\rangle_0 \\
&= \langle -iA_1, -jA_3\rangle_0 + \langle -A_1, kA_3\rangle_0 \\
&= 0 = -\gamma_0(I, I, K) = \mathrm{d}\gamma_0(1, i, I, K) \\
\frac{1}{2}\langle \alpha_0 \wedge \alpha_0\rangle(1, i, J, K) &= \langle \alpha_0(J, 1), \alpha_0(i, K)\rangle_0 + \langle \alpha_0(i, J), \alpha_0(1, K)\rangle_0 \\
&= \langle -jA_2, -jA_3\rangle_0 + \langle kA_2, kA_3\rangle_0 \\
&= -2 = -\gamma_0(I, J, K) = \mathrm{d}\gamma_0(1, i, J, K).
\end{aligned}
$$

It is easy to see that (α_0, γ_0) is admissible. Indeed, Condition (T) is satisfied because of $(\mathfrak{a}_0)_+ = 0$ and (B_0) and (B_1) hold since $\langle\cdot\,,\cdot\rangle_0$ is definite. As for Conditions (A_0) and (A_1) we observe that $\mathfrak{z}(\mathfrak{l}_0) \cap \mathfrak{l}_0 = \mathfrak{z}(\mathfrak{l}_0) \cap \mathfrak{l}_0' = \mathrm{span}\,\{I, J, K\}$ and that for all $Q \in \mathrm{span}\,\{I, J, K\}$ there is an element $L \in \mathfrak{l}_0$ such that $\alpha_0(L, Q) \neq 0$. Hence, (A_0) and (A_1) are also satisfied, thus (α_0, γ_0) is admissible. Obviously, (α_0, γ_0) is also indecomposable since $\mathfrak{l}_0$ is indecomposable and $\alpha_0(\mathfrak{l}_0, \mathfrak{l}_0) = \mathfrak{a}_0$. □

Corollary 5.2. *The triple $\mathfrak{d}_{\alpha_0,\gamma_0}(\mathfrak{l}_0, \Phi_{\mathfrak{l}_0}, \mathfrak{a}_0)$ is an indecomposable hyper-Kähler symmetric triple of signature (* 4, 12*). The holonomy group of a symmetric space associated with this triple is non-abelian.*

Proof. The first statement follows from Proposition 4.1, Lemma 5.1 and Eq. (5). It remains to prove the assertion on the holonomy group. The Lie algebra of this group is isomorphic to $\mathfrak{d}_+ = (\mathfrak{l}_0^*)_+ \oplus (\mathfrak{a}_0)_+ \oplus (\mathfrak{l}_0)_+$ with Lie bracket defined by (2)–(4). Since $\gamma(I, J, K) \neq 0$ this Lie algebra is not abelian. □

Remark 5.3. Let us briefly recall how Alekseevsky and Cortés [2] describe hyper-Kähler symmetric triples. Let (E, ω) be a complex symplectic vector space. Any $S \in S^4E$ defines a complex linear subspace $\mathfrak{h}_S \subset \mathfrak{sp}(E, \omega) \cong S^2E$ by

$$\mathfrak{h}_S = \mathrm{span}\,\{S_{v,w} \in S^2E \mid v, w \in E\},$$

where $S_{v,w}$ is the contraction of S with v and w defined by the symplectic form ω. If

$$S \in (S^4E)^{\mathfrak{h}_S}, \tag{7}$$

then $\mathfrak{h}_S \subset \mathfrak{sp}(E, \omega)$ is a Lie subalgebra and, moreover, there is a natural Lie bracket on $\mathfrak{h}_S \oplus (\mathbb{H} \otimes_{\mathbb{C}} E)$. If there exists a Lagrangian subspace $E_+ \subset E$ such that $S \in S^4E_+ \subset S^4E$, then S is a solution of (7). Let us call solutions of this kind tame. If S is tame, then the Lie algebra $\mathfrak{h}_S$ is abelian.

Let J be a quaternionic structure on E such that $J^*\omega = \bar{\omega}$. Then J induces real structures, all denoted by τ, on S^4E, $S^2E \cong \mathfrak{sp}(E, \omega)$, and on $\mathbb{H} \otimes_{\mathbb{C}} E$. If $S \in (S^4E)^\tau$ satisfies (7), then the real Lie algebra

$$\mathfrak{g}_S := (\mathfrak{h}_S)^\tau \oplus (\mathbb{H} \otimes_{\mathbb{C}} E)^\tau$$

carries a canonical structure of a hyper-Kähler symmetric triple. Moreover, Alekseevsky and Cortés proved that all hyper-Kähler symmetric triples arise in this way. In fact, they describe explicitly how to find the polynomial S for a given hyper-Kähler symmetric triple.

Using this description it is not difficult to show that $\mathfrak{d}_{\alpha_0,\gamma_0}(\mathfrak{l}_0, \Phi_{\mathfrak{l}_0}, \mathfrak{a}_0) \cong \mathfrak{g}_S$ for (E, ω, J, S) as follows: (E, ω) is 8-dimensional with standard symplectic basis $p_1, \ldots, p_4, q_1, \ldots, q_4$, the quaternionic structure J is characterized by $J(p_1) = p_2$, $J(p_i) = q_i$ for $i = 3, 4$, and

$$S = p_1^3 q_3 + \sqrt{3}p_1^2 p_2 p_4 - \sqrt{3}p_1 p_2^2 q_4 - p_2^3 p_3.$$

Let us verify directly that S satisfies (7). Using the τ-invariance of S we find that $\mathfrak{h}_S = V + \tau(V)$, where

$$
\begin{aligned}
V &= \mathrm{span}\,\{S_{q_1,q_1},\ S_{q_1,q_2},\ S_{q_1,q_4},\ S_{q_1,p_3},\ S_{q_1,p_4}\} \\
&= \mathrm{span}\,\{\sqrt{3}p_1 q_3 + p_2 p_4,\ p_1 p_4 - p_2 q_4,\ p_1 p_2,\ p_1^2,\ p_2^2\}.
\end{aligned}
$$

It suffices to check that $P(S) = 0$ for P running through the above set of basis elements of V. Since $S_{p_1} = S_{p_2} = 0$ we immediately see that $P(S) = 0$ if P is one of the last three basis elements. It remains to compute

$$(\sqrt{3}p_1q_3 + p_2p_4)(S) = 2(\sqrt{3}p_1S_{q_3} + p_2S_{p_4}) = \frac{1}{2}(\sqrt{3}p_1p_2^3 + p_2(-\sqrt{3}p_1p_2^2)) = 0,$$
$$(p_1p_4 - p_2q_4)(S) = 2(p_1S_{p_4} - p_2S_{q_4}) = \frac{1}{2}(p_1(-\sqrt{3}p_1p_2^2) - p_2(-\sqrt{3}p_1^2p_2)) = 0.$$

Eq. (7) follows. On the other hand, S is not tame since $\{v \in E \mid S_v = 0\} = \mathrm{span}\,\{p_1, p_2\}$ is only two-dimensional. This shows that Theorem 3 in [2] claiming that all solutions of (7) are tame is not true. In particular, those results of [2,1] which are based on this theorem have to be reconsidered. Note however, that a recent result of Cortés ([5], Theorem 10; compare also [6]) says that all hyper-Kähler symmetric triples with abelian $\mathfrak{g}_+$ are given by tame solutions of (7). Up to now, there is no general method to find all solutions of (7). It is not even clear how to find the special solution desribed above without knowing the symmetric triple in advance.

Example 2

We look at $\mathbb{H}^2$ as an abelian Lie algebra equipped with the quaternionic grading given by the left $\mathbb{H}$-module structure. We equip $\mathrm{Im}\,\mathbb{H} \oplus \mathrm{Im}\,\mathbb{H}$ with the trivial $Sp(1)$-action. Let $A : \mathrm{Im}\,\mathbb{H} \to \mathrm{Im}\,\mathbb{H}$ be a real linear traceless and bijective map which is symmetric with respect to the standard scalar product $\langle\cdot\,,\cdot\rangle_{\mathbb{H}}|_{\mathrm{Im}\,\mathbb{H}}$. Then we define an inner product on $\mathrm{Im}\,\mathbb{H} \oplus \mathrm{Im}\,\mathbb{H}$ by

$$\langle (P_1, P_2), (Q_1, Q_2)\rangle := \langle P_1, AQ_2\rangle_{\mathbb{H}} + \langle Q_1, AP_2\rangle_{\mathbb{H}}$$

and denote the resulting vector space with orthogonal quaternionic grading by $\mathfrak{a}_A$. We define $\alpha_+ \in C^2(\mathbb{H}^2, \mathfrak{a}_A)^{Sp(1)}$ by

$$\alpha_+((p, q), (r, s)) := (\mathrm{Im}(\bar{p}s + \bar{q}r), \mathrm{Im}(\bar{q}s)).$$

We claim that

$$(\alpha_+, 0) \in \mathcal{Z}^2_Q(\mathbb{H}^2, \Phi_{\mathbb{H}^2}, \mathfrak{a}_A). \tag{8}$$

We have to check that $\langle \alpha_+ \wedge \alpha_+\rangle = 0$. We decompose $\bigwedge^4 \mathbb{H}^2 = \bigoplus_{k+l=4} \bigwedge^{k,l}$, where $\bigwedge^{k,l} = \bigwedge^k \mathbb{H} \otimes \bigwedge^l \mathbb{H}$. First we observe that $\langle \alpha_+ \wedge \alpha_+\rangle$ vanishes on $\bigwedge^{k,l}$ for $(k, l) \neq (1, 3)$. Next we compute

$$\begin{aligned}\frac{1}{2}\langle \alpha_+ \wedge \alpha_+\rangle((1, 0), (0, i), (0, j), (0, k)) &= \langle i, A(-i)\rangle_{\mathbb{H}} + \langle k, A(-k)\rangle_{\mathbb{H}} + \langle -j, A(j)\rangle_{\mathbb{H}}\\ &= -\mathrm{tr}\, A = 0,\end{aligned}$$

and for $p, q \in \mathrm{Im}\,\mathbb{H} \subset \mathbb{H}$

$$\frac{1}{2}\langle \alpha_+ \wedge \alpha_+\rangle((1, 0), (0, p), (0, q), (0, 1)) = \langle p, A(-q)\rangle_{\mathbb{H}} + \langle -q, A(-p)\rangle_{\mathbb{H}} = 0.$$

$Sp(1)$-invariance implies that $\langle \alpha_+ \wedge \alpha_+\rangle$ vanishes on $\bigwedge^{1,3}$ as well. This proves (8).

Let $\mathfrak{l}_0, \mathfrak{a}_0, \alpha_0, \gamma_0$ be as in Example 1. Fix $n \in \mathbb{N}$ and choose a vector $\mathfrak{A} = (A_1, \ldots, A_n)$ of traceless symmetric bijective maps $A_k : \mathrm{Im}\,\mathbb{H} \to \mathrm{Im}\,\mathbb{H}$. We equip

$$\mathfrak{l} := \mathfrak{l}_0 \oplus \mathbb{H}^n, \qquad \mathfrak{a}_{\mathfrak{A}} := \mathfrak{a}_0 \oplus \bigoplus_{k=1}^{n} \mathfrak{a}_{A_k}$$

with their natural quaternionic gradings. Then we have natural projections

$$\varphi_0 : \mathfrak{l} \longrightarrow \mathfrak{l}_0$$

and for $k = 1, \ldots, n$

$$\varphi_k : \mathfrak{l} \longrightarrow \mathbb{H}^2, \qquad \varphi_k(p, P, p_1, \ldots, p_n) := (p, p_k).$$

We define

$$\alpha := \varphi_0^*\alpha_0 \oplus \bigoplus_{k=1}^{n} \varphi_k^*\alpha_+ \in C^2(\mathfrak{l}, \mathfrak{a}_{\mathfrak{A}})^{Sp(1)}, \qquad \gamma := \varphi_0^*\gamma_0 \in C^3(\mathfrak{l}).$$

Combining (8) with Lemma 5.1 we see that $(\alpha, \gamma) \in \mathcal{Z}^2_Q(\mathfrak{l}, \Phi_{\mathfrak{l}}, \mathfrak{a}_{\mathfrak{A}})$.

Lemma 5.4. *The cocycle* $(\alpha, \gamma) \in \mathcal{Z}^2_Q(\mathfrak{l}, \Phi_{\mathfrak{l}}, \mathfrak{a}_{\mathfrak{A}})$ *is admissible and indecomposable.*

Proof. A straightforward verification shows that (α, γ) satisfies the admissibility conditions of Definition 3.1. In particular, α enjoys the following properties

(a) $\alpha(\bigwedge^2 \mathfrak{l}) = \mathfrak{a}$,
(b) For all $L \in \mathfrak{l} \setminus \{0\}$ there exists $L' \in \mathfrak{l}$ such that $\alpha(L, L') \neq 0$.

Now we use Properties (a) and (b) in order to show indecomposability of (α, γ). Assume that we have decompositions

$$\mathfrak{l} = \mathfrak{l}_1 \oplus \mathfrak{l}_2, \qquad \mathfrak{a} = \mathfrak{a}_1 \oplus \mathfrak{a}_2, \qquad \alpha = \mathrm{pr}_1^*\alpha_1 \oplus \mathrm{pr}_2^*\alpha_2,$$

for certain $\alpha_i \in C^2(\mathfrak{l}_i, \mathfrak{a}_i)^{Sp(1)}$. We may assume that $\mathfrak{l}_1$ is non-abelian. Then $Sp(1)$-invariance of $\mathfrak{l}_1$ implies that $\mathfrak{l}_+ = (\mathfrak{l}_0)_+ = (\mathfrak{l}_1)_+$. This forces $(\mathfrak{l}_2)_+ = [(\mathfrak{l}_2)_-, (\mathfrak{l}_2)_-] = 0$, which in turn implies $\mathfrak{l}_2 \subset \mathbb{H}^n$. It follows that $\mathfrak{a}_2 = \alpha(\bigwedge^2 \mathfrak{l}_2)$ is isotropic, hence $\mathfrak{a}_2 = \{0\}$, $\alpha_2 = 0$. Therefore $\alpha = \mathrm{pr}_1^*\alpha_1$. Now (b) implies that $\mathfrak{l}_2 = \{0\}$. It follows that (α, γ) is indecomposable. □

As in Example 1 we obtain

Corollary 5.5. $\mathfrak{d}_{\alpha,\gamma}(\mathfrak{l}, \Phi_{\mathfrak{l}}, \mathfrak{a}_{\mathfrak{A}})$ *is an indecomposable hyper-Kähler symmetric triple of signature* $(4n + 4, 4n + 12)$. *The holonomy group of a symmetric space associated with this triple is non-abelian.*

Example 3

Let $(\mathfrak{l}, \Phi_{\mathfrak{l}})$ be a Lie algebra with proper quaternionic grading. Then we can form the cotangent Lie algebra $T^*\mathfrak{l} := \mathfrak{l} \ltimes \mathfrak{l}^*$ which possesses a quaternionic grading $T^*\Phi_{\mathfrak{l}}$ and an invariant metric $\langle \cdot, \cdot \rangle_{T^*\mathfrak{l}}$ (given by the dual pairing) in a natural way. This construction could be viewed as a special case of the one presented in Section 4 — apart from the fact that the involved cocycle (0, 0) is not admissible. If we require in addition that

$$\mathfrak{z}(\mathfrak{l}) \subset \mathfrak{l}_-, \tag{9}$$

then $(T^*\mathfrak{l}, T^*\Phi_{\mathfrak{l}}, \langle \cdot, \cdot \rangle_{T^*\mathfrak{l}})$ is a hyper-Kähler symmetric triple.

If $(\mathfrak{g}, \Phi_{\mathfrak{g}}, \langle \cdot, \cdot \rangle_{\mathfrak{g}})$ is a hyper-Kähler symmetric triple, then $\mathfrak{g}$ satisfies (9), and the cotangent hyper-Kähler symmetric triple $(T^*\mathfrak{g}, T^*\Phi_{\mathfrak{g}}, \langle \cdot, \cdot \rangle_{T^*\mathfrak{g}})$ is isomorphic to the tangent triple $(T\mathfrak{g}, T\Phi_{\mathfrak{g}}, \langle \cdot, \cdot \rangle_{T\mathfrak{g}})$, where

$$T\mathfrak{g} := \mathfrak{g} \ltimes \mathfrak{g}, \quad \langle (X_1, Y_1), (X_2, Y_2) \rangle_{T\mathfrak{g}} := \langle X_1, Y_2 \rangle_{\mathfrak{g}} + \langle X_2, Y_1 \rangle_{\mathfrak{g}}.$$

Proposition 5.6. (a) *Let* $(\mathfrak{l}, \Phi_{\mathfrak{l}})$ *be a non-abelian Lie algebra with proper quaternionic grading satisfying* (9). *If* $(\mathfrak{l}, \Phi_{\mathfrak{l}})$ *is indecomposable as a Lie algebra with quaternionic grading, then the hyper-Kähler symmetric triple* $(T^*\mathfrak{l}, T^*\Phi_{\mathfrak{l}}, \langle \cdot, \cdot \rangle_{T^*\mathfrak{l}})$ *is indecomposable.*
(b) *Let* $(\mathfrak{g}, \Phi_{\mathfrak{g}}, \langle \cdot, \cdot \rangle_{\mathfrak{g}})$ *be an indecomposable hyper-Kähler symmetric triple with* $\mathfrak{g}$ *non-abelian. Then the hyper-Kähler symmetric triple* $(T\mathfrak{g}, T\Phi_{\mathfrak{g}}, \langle \cdot, \cdot \rangle_{T\mathfrak{g}})$ *is indecomposable as well.*

Proof. Let $(\mathfrak{g}, \Phi_{\mathfrak{g}}, \langle \cdot, \cdot \rangle_{\mathfrak{g}})$ be an indecomposable hyper-Kähler symmetric triple with $\mathfrak{g}$ non-abelian. By Lemma 2.5 the tuple $(\mathfrak{g}, \langle \cdot, \cdot \rangle_{\mathfrak{g}})$ is indecomposable as a metric Lie algebra. Then, according to [3], Theorem 5, the metric Lie algebra $(T\mathfrak{g}, \langle \cdot, \cdot \rangle_{T\mathfrak{g}})$ is indecomposable as well. This implies (b).

Now let $(\mathfrak{l}, \Phi_{\mathfrak{l}})$ be as in (a). We consider the Lie algebra $\mathfrak{sp}(1) \ltimes \mathfrak{l}$, where $\mathfrak{sp}(1)$ acts on $\mathfrak{l}$ by the differential of $\Phi_{\mathfrak{l}}$. We let $\mathfrak{sp}(1) \ltimes \mathfrak{l}$ act on $T^*\mathfrak{l}$ by $(Q, L)v := d(T^*\Phi_{\mathfrak{l}})(Q)v + [L, v]$, $Q \in \mathfrak{sp}(1)$, $L \in \mathfrak{l}$, $v \in T^*\mathfrak{l}$. Then $T^*\mathfrak{l} = \mathfrak{l} \oplus \mathfrak{l}^*$ is a decomposition into indecomposable $\mathfrak{sp}(1) \ltimes \mathfrak{l}$-submodules. Let us assume that there is a non-trivial decomposition into hyper-Kähler symmetric triples $T^*\mathfrak{l} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$. In particular, $\mathfrak{h}_1$ and $\mathfrak{h}_2$ are $\mathfrak{sp}(1) \ltimes \mathfrak{l}$-submodules. The Krull-Schmidt Theorem (see e.g. [7], p. 115) implies that one of these modules, say $\mathfrak{h}_1$, is isomorphic to $\mathfrak{l}$. Pulling back the invariant bilinear form $\langle\cdot,\cdot\rangle_{T^*\mathfrak{l}}|_{\mathfrak{h}_1}$ from $\mathfrak{h}_1$ to $\mathfrak{l}$ we obtain an inner product $\langle\cdot,\cdot\rangle_{\mathfrak{l}}$ such that $(\mathfrak{l}, \Phi_{\mathfrak{l}}, \langle\cdot,\cdot\rangle_{\mathfrak{l}})$ is an indecomposable hyper-Kähler symmetric triple. Now we apply (b) to this triple which yields a contradiction to the assumption that $T^*\mathfrak{l} \cong T\mathfrak{l}$ is decomposable as a hyper-Kähler symmetric triple. This finishes the proof of (a). □

If the symbol $\mathfrak{d}$ stands for a hyper-Kähler symmetric triple, then we simply write $T\mathfrak{d}$ or $T^1\mathfrak{d}$ for its tangential triple. We form higher tangential triples by $T^n\mathfrak{d} := T(T^{n-1}\mathfrak{d})$, $n \geq 2$.

Corollary 5.7. *Let $\mathfrak{d}$ be one of the hyper-Kähler symmetric triples constructed in Examples 1 and 2 of signature $(4k, 4k+8)$, $k \geq 1$. Then $T^n\mathfrak{d}$, $n \geq 1$, is an indecomposable hyper-Kähler symmetric triple of neutral signature $(2^{n+2}(k+1), 2^{n+2}(k+1))$. The holonomy group of a symmetric space associated with $T^n\mathfrak{d}$ is non-abelian.*

Remark 5.8. If $\mathfrak{d} = \mathfrak{d}_{\alpha,\gamma}(\mathfrak{l}, \Phi_{\mathfrak{l}}, \mathfrak{a})$ for some admissible cocycle $(\alpha, \gamma) \in \mathcal{Z}^2_Q(\mathfrak{l}, \Phi_{\mathfrak{l}}, \mathfrak{a})$ (as e.g. in Corollary 5.7), then $T\mathfrak{d} \cong \mathfrak{d}_{T\alpha,T\gamma}(T\mathfrak{l}, T\Phi_{\mathfrak{l}}, T\mathfrak{a})$, where

$$T\alpha\left((L_1, L_1'), (L_2, L_2')\right) = \left(\alpha(L_1, L_2), \alpha(L_1, L_2') + \alpha(L_1', L_2)\right),$$
$$T\gamma\left((L_1, L_1'), (L_2, L_2'), (L_3, L_3')\right) = \gamma(L_1, L_2, L_3') + \gamma(L_1, L_2', L_3) + \gamma(L_1', L_2, L_3).$$

Moreover, one can show that the cocycle $(T\alpha, T\gamma)$ is admissible again. This last assertion is not true for the general symmetric triples $\mathfrak{d}_{\alpha,\gamma}(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a})$ constructed in [9] from admissible cocycles $(\alpha, \gamma) \in \mathcal{Z}^2_Q(\mathfrak{l}, \theta_{\mathfrak{l}}, \mathfrak{a})$, only for those with nilpotent $\mathfrak{l}$ and a trivial $\mathfrak{l}$-module $\mathfrak{a}$.

6. A classification scheme

Let $(\mathfrak{l}, \Phi_{\mathfrak{l}})$ be a Lie algebra with proper quaternionic grading and let $(\mathfrak{a}, \langle\cdot,\cdot\rangle_{\mathfrak{a}}, \Phi_{\mathfrak{a}})$ be a vector space with orthogonal quaternionic grading. We consider their automorphism groups $\mathrm{Aut}\,(\mathfrak{l}, \Phi_{\mathfrak{l}})$, and $\mathrm{Aut}\,(\mathfrak{a}, \langle\cdot,\cdot\rangle_{\mathfrak{a}}, \Phi_{\mathfrak{a}})$ and form

$$G_{\mathfrak{l},\Phi_{\mathfrak{l}},\mathfrak{a}} := \mathrm{Aut}\,(\mathfrak{l}, \Phi_{\mathfrak{l}}) \times \mathrm{Aut}\,(\mathfrak{a}, \langle\cdot,\cdot\rangle_{\mathfrak{a}}, \Phi_{\mathfrak{a}}).$$

There is a natural right action of $G_{\mathfrak{l},\Phi_{\mathfrak{l}},\mathfrak{a}}$ on $\mathcal{H}^2_Q(\mathfrak{l}, \Phi_{\mathfrak{l}}, \mathfrak{a})$ which leaves $\mathcal{H}^2_Q(\mathfrak{l}, \Phi_{\mathfrak{l}}, \mathfrak{a})_0$ invariant.

Theorem 6.1. *There is a bijective map from the set of isomorphism classes of indecomposable hyper-Kähler symmetric triples to the union of orbit spaces*

$$\coprod_{(\mathfrak{l},\Phi_{\mathfrak{l}})} \coprod_{(\mathfrak{a},\langle\cdot,\cdot\rangle_{\mathfrak{a}},\Phi_{\mathfrak{a}})} \mathcal{H}^2_Q(\mathfrak{l}, \Phi_{\mathfrak{l}}, \mathfrak{a})_0 / G_{\mathfrak{l},\Phi_{\mathfrak{l}},\mathfrak{a}},$$

where the union is taken over all isomorphism classes of Lie algebras $(\mathfrak{l}, \Phi_{\mathfrak{l}})$ with proper quaternionic grading and all isomorphism classes of vector spaces $(\mathfrak{a}, \langle\cdot,\cdot\rangle_{\mathfrak{a}}, \Phi_{\mathfrak{a}})$ with orthogonal quaternionic grading.

The inverse of this map sends the orbit of $[\alpha, \gamma] \in \mathcal{H}^2_Q(\mathfrak{l}, \Phi_{\mathfrak{l}}, \mathfrak{a})_0$ to the isomorphism class of $\mathfrak{d}_{\alpha,\gamma}(\mathfrak{l}, \Phi_{\mathfrak{l}}, \mathfrak{a})$.

Proof. The theorem is the hyper-Kähler analog of our classification scheme of symmetric triples [9], Theorem 6.1. The proof of the latter is contained in [9], Sections 4–6. It carries over to the present situation, one has to take care of the quaternionic gradings, only. For the convenience of the reader we describe the map which associates a cohomology class to a given hyper-Kähler symmetric triple $(\mathfrak{g}, \Phi_{\mathfrak{g}}, \langle\cdot,\cdot\rangle_{\mathfrak{g}})$.

By Corollary 2.4 the Lie algebra $\mathfrak{g}$ is nilpotent. We look at its lower central series and form the isotropic ideal

$$\mathfrak{i} := \sum_{k=2}^{\infty} \mathfrak{g}^k \cap (\mathfrak{g}^k)^{\perp}. \tag{10}$$

Then $\mathfrak{l} := \mathfrak{g}/\mathfrak{i}^{\perp}$ and $\mathfrak{a} := \mathfrak{i}^{\perp}/\mathfrak{i}$ inherit the desired structures from $\mathfrak{g}$. Moreover, the nilpotency of $\mathfrak{g}$ implies that the induced $\mathfrak{l}$-action on $\mathfrak{a}$ is trivial. We choose an $Sp(1)$-equivariant section $s : \mathfrak{l} \to \mathfrak{g}$ with isotropic image and set

$$\alpha(L_1, L_2) := [s(L_1), s(L_2)]_{\mathfrak{g}} - s([L_1, L_2]_{\mathfrak{l}}) \bmod \mathfrak{i}$$
$$\gamma(L_1, L_2, L_3) := \langle [s(L_1), s(L_2)]_{\mathfrak{g}}, s(L_3)\rangle_{\mathfrak{g}}.$$

Then (α, γ) is an admissible cocycle in $\mathcal{Z}^2_Q(\mathfrak{l}, \Phi_\mathfrak{l}, \mathfrak{a})$. It is indecomposable if $(\mathfrak{g}, \Phi_\mathfrak{g}, \langle\cdot,\cdot\rangle_\mathfrak{g})$ is so. The desired cohomology class is given by $[\alpha, \gamma] \in \mathcal{H}^2_Q(\mathfrak{l}, \Phi_\mathfrak{l}, \mathfrak{a})_0$. □

Remark 6.2. For $[\alpha, \gamma] \in \mathcal{H}^2_Q(\mathfrak{l}, \Phi_\mathfrak{l}, \mathfrak{a})_0$ a hyper-Kähler symmetric space associated with $\mathfrak{d}_{\alpha,\gamma}(\mathfrak{l}, \Phi_\mathfrak{l}, \mathfrak{a})$ has abelian holonomy if and only if $\mathfrak{l}$ is abelian. Indeed, if $\mathfrak{l}$ is abelian, then $\mathfrak{l} = \mathfrak{l}_-$, since $\mathfrak{l}$ is proper and the assertion follows from Eqs. (2)–(4). To show the converse, recall from Remark 5.3 that Cortés [5] showed that every hyper-Kähler symmetric triple $(\mathfrak{g}, \Phi, \langle\cdot,\cdot\rangle)$ with abelian $\mathfrak{g}_+$ arises from a tame solution of (7). Now using (10) it is easy to see that $\mathfrak{l} = \mathfrak{g}/\mathfrak{i}^\perp$ is abelian.

7. Classification in the case of signature (4, 4*n*)

Our general classification scheme can be used to find explicit classification results (i.e. lists), if we only consider hyper-Kähler symmetric spaces of a given small index. Here we will classify indecomposable hyper-Kähler symmetric spaces of index 4. We want to apply Theorem 6.1. So we have to decide for which $(\mathfrak{l}, \Phi_\mathfrak{l})$ there is a vector space with orthogonal quaternionic grading $\mathfrak{a}$ and an element $[\alpha, \gamma] \in \mathcal{H}^2_Q(\mathfrak{l}, \Phi_\mathfrak{l}, \mathfrak{a})_0$ such that $\mathfrak{d}_{\alpha,\gamma}(\mathfrak{l}, \Phi_\mathfrak{l}, \mathfrak{a})$ has index 4.

Proposition 7.1. *Let $(\mathfrak{l}, \Phi_\mathfrak{l})$ be a Lie algebra with proper quaternionic grading, let $\mathfrak{a}$ be a vector space with orthogonal quaternionic grading and suppose $[\alpha, \gamma] \in \mathcal{H}^2_Q(\mathfrak{l}, \Phi_\mathfrak{l}, \mathfrak{a})_0$. If $\mathfrak{d}_{\alpha,\gamma}(\mathfrak{l}, \Phi_\mathfrak{l}, \mathfrak{a})$ is a hyper-Kähler symmetric triple of index 4, then $\mathfrak{l} = 0$ or $(\mathfrak{l}, \Phi_\mathfrak{l})$ is isomorphic either to $(\mathfrak{l}_0, \Phi_{\mathfrak{l}_0})$ (as defined in Section 5) or to $(\mathbb{H}, \Phi_\mathbb{H})$, where $\Phi_\mathbb{H}$ is the left multiplication on $\mathbb{H}$.*

Proof. From (5) we know that $\dim \mathfrak{l}_- \in \{0, 4\}$. If $\dim \mathfrak{l}_- = 0$, then $\mathfrak{l} = 0$. Suppose now that $\dim \mathfrak{l}_- = 4$, i.e. $\mathfrak{l}_- \cong \mathbb{H}$. Consider now the map $\mathrm{ad}\, L|_{\mathfrak{l}_-}$ for an arbitrary $L \in \mathfrak{l}_+$. This map commutes with the $Sp(1)$-action, i.e. it is $\mathbb{H}$-linear. On the other hand it must be nilpotent, since $\mathfrak{l}$ is nilpotent by Proposition 2.1. Hence the map is zero, and it follows that $[\mathfrak{l}_-, \mathfrak{l}_+] = 0$. This gives $[\mathfrak{l}_+, \mathfrak{l}_+] = [[\mathfrak{l}_-, \mathfrak{l}_-], \mathfrak{l}_+] = 0$. In particular, we have $\mathfrak{l}' = \mathfrak{l}_+ \subset \mathfrak{z}(\mathfrak{l})$.

Since $Sp(1)$ acts by automorphisms we have

$$[1, i] = -[j, k], \qquad [1, j] = [i, k], \qquad [1, k] = -[i, j].$$

In particular we get $\dim \mathfrak{l}' \le 3$. If $\dim \mathfrak{l}' = 0$, then $\mathfrak{l} \cong \mathbb{H}$, if $\dim \mathfrak{l}' = 3$, then $\mathfrak{l} \cong \mathfrak{l}_0$. Assume $\dim \mathfrak{l}' = 1$. By an easy direct computation or using [8], Proposition 6.2 one obtains $\mathcal{H}^2_Q(\mathfrak{l}, \Phi_\mathfrak{l}, \mathfrak{a})_\sharp = \emptyset$, a contradiction.

It remains to exclude the case $\dim \mathfrak{l}' = 2$. If $\dim \mathfrak{l}' = 2$, then we may assume

$$[1, i] = -[j, k] =: X, \qquad [1, j] = [i, k] =: Y, \qquad [1, k] = -[i, j] = 0.$$

Then $d\alpha = 0$ implies $\alpha(X, Y) = 0$ and $0 = d\alpha(1, i, j) = \alpha([1, i], j) + \alpha([j, 1], i)$, thus $\alpha(X, j) = \alpha(Y, i) =: A$. This together with the $Sp(1)$-invariance of α implies $\mathfrak{a}_- = \alpha(\mathfrak{l}_-, \mathfrak{l}_+) = \mathrm{span}\,\{A, iA, jA, kA\}$. But $d\gamma(i, j, X, Y) = 0$ now implies

$$0 = \langle \alpha(j, X), \alpha(i, Y)\rangle + \langle \alpha(X, i), \alpha(j, Y)\rangle = 2\langle A, A\rangle.$$

Hence A is isotropic. Since $\mathfrak{a}_-$ is non-degenerate it follows that $A = 0$, thus $\alpha(\mathfrak{l}_-, \mathfrak{l}_+) = 0$. In particular, $\alpha(X, \mathfrak{l}) = 0$. Hence α satisfies Condition (i) of (A_1) for $L_0 = X$. Let us show that Condition (ii) is also satisfied. Because of the $Sp(1)$-invariance of γ we have $\gamma(\mathfrak{l}_+, \mathfrak{l}_+, \mathfrak{l}_-) = 0$. Since, moreover, $\dim \mathfrak{l}_+ = 2$ we obtain $\gamma(\mathfrak{l}_+, \mathfrak{l}_+, \mathfrak{l}) = 0$. In particular, $\gamma(L, X, \cdot) = 0 \in (\mathfrak{l}')^*$ for all $L \in \mathfrak{l}$. Hence (ii) is satisfied for $A_0 = 0$ and $Z_0 = 0$. Now (A_1) implies $X = 0$, a contradiction. □

Next we have to determine $\mathcal{H}^2_Q(\mathfrak{l}, \Phi_\mathfrak{l}, \mathfrak{a})_0 / G_{\mathfrak{l},\Phi_\mathfrak{l},\mathfrak{a}}$ for all $(\mathfrak{l}, \Phi_\mathfrak{l})$ which appear in Proposition 7.1 and for all (suitable) $\mathfrak{a}$. The case $\mathfrak{l} = 0$ is trivial. Thus let us start with $(\mathfrak{l}, \Phi_\mathfrak{l}) = (\mathbb{H}, \Phi_\mathbb{H})$. For a fixed orthonormal basis A_1, A_2 of $\mathfrak{a} = \mathfrak{a}_+ = \mathbb{R}^{1,1}$ we define $\alpha' \in Z^2(\mathbb{H}, \mathfrak{a})^{Sp(1)}$ by

$$\alpha'(1, i) = A_1, \qquad \alpha'(1, j) = A_2, \qquad \alpha'(1, k) = 0.$$

Furthermore, for a fixed orthonormal basis A_1, A_2, A_3 of $\mathfrak{a} = \mathfrak{a}_+ = \mathbb{R}^{1,2}$ and each real number $0 < r \le \pi/4$ we define $\alpha_r \in Z^2(\mathbb{H}, \mathfrak{a})^{Sp(1)}$ by

$$\alpha_r(1, i) = A_1, \qquad \alpha_r(1, j) = \sin r \cdot A_2, \qquad \alpha_r(1, k) = \cos r \cdot A_3.$$

Analogously, for a fixed orthonormal basis A_1, A_2, A_3 of $\mathfrak{a} = \mathfrak{a}_+ = \mathbb{R}^{2,1}$ and each real number $0 < s \le \pi/4$ we define $\alpha_s \in Z^2(\mathbb{H}, \mathfrak{a})^{Sp(1)}$ by

$$\alpha_s(1,i) = \sin s \cdot A_1, \qquad \alpha_s(1,j) = \cos s \cdot A_2, \qquad \alpha_s(1,k) = A_3.$$

Then it is easy to verify that $\langle \alpha' \wedge \alpha' \rangle = 0$, $\langle \alpha_r \wedge \alpha_r \rangle = 0$, and $\langle \alpha_s \wedge \alpha_s \rangle = 0$ holds, e.g.

$$\begin{aligned}\langle \alpha_r \wedge \alpha_r \rangle(1,i,j,k) &= \langle \alpha_r(1,i), \alpha_r(j,k)\rangle + \langle \alpha_r(j,1), \alpha_r(i,k)\rangle + \langle \alpha_r(i,j), \alpha_r(1,k)\rangle \\ &= \langle A_1, -A_1\rangle + \langle -\sin r \cdot A_2, \sin r \cdot A_2\rangle + \langle -\cos r \cdot A_3, \cos r \cdot A_3\rangle \\ &= 1 - \sin^2 r - \cos^2 r = 0.\end{aligned}$$

Proposition 7.2. *We equip the spaces* $\mathbb{R}^{1,1}$, $\mathbb{R}^{1,2}$ *and* $\mathbb{R}^{2,1}$ *with the trivial* $Sp(1)$*-action. Then we have*

$$\begin{aligned}&\mathcal{H}^2_Q(\mathbb{H}, \Phi_{\mathbb{H}}, \mathbb{R}^{1,1})_0 / G_{\mathbb{H},\Phi_{\mathbb{H}},\mathbb{R}^{1,1}} = \{[\alpha', 0]\}\\ &\mathcal{H}^2_Q(\mathbb{H}, \Phi_{\mathbb{H}}, \mathbb{R}^{1,2})_0 / G_{\mathbb{H},\Phi_{\mathbb{H}},\mathbb{R}^{1,2}} = \{[\alpha_r, 0] \mid 0 < r \le \pi/4\}\\ &\mathcal{H}^2_Q(\mathbb{H}, \Phi_{\mathbb{H}}, \mathbb{R}^{2,1})_0 / G_{\mathbb{H},\Phi_{\mathbb{H}},\mathbb{R}^{2,1}} = \{[\alpha_s, 0] \mid 0 < s \le \pi/4\}.\end{aligned}$$

If $\mathfrak{a}$ *is a vector space with orthogonal quaternionic grading which is not isomorphic to* $\mathbb{R}^{1,1}$, $\mathbb{R}^{1,2}$ *or* $\mathbb{R}^{2,1}$ *with the trivial* $Sp(1)$*-action, then* $\mathcal{H}^2_Q(\mathbb{H}, \Phi_{\mathbb{H}}, \mathfrak{a})_0$ *is empty.*

Proof. Let $\mathfrak{a}$ be a vector space with orthogonal quaternionic grading. Since $C^3(\mathbb{H})^{Sp(1)}$ vanishes and $H^2(\mathbb{H}, \mathfrak{a}) = C^2(\mathbb{H}, \mathfrak{a})$ we have

$$\mathcal{H}^2_Q(\mathbb{H}, \Phi_{\mathbb{H}}, \mathfrak{a}) = \{[\alpha, 0] \mid \alpha \in C^2(\mathbb{H}, \mathfrak{a})^{Sp(1)}, \langle \alpha \wedge \alpha \rangle = 0\}.$$

Each element $\alpha \in C^2(\mathbb{H}, \mathfrak{a})^{Sp(1)}$ is given by the values

$$\alpha(1,i) = A'_1, \qquad \alpha(1,j) = A'_2, \qquad \alpha(1,k) = A'_3, \quad A'_i \in \mathfrak{a}_+, \tag{11}$$

and $\langle \alpha \wedge \alpha \rangle = 0$ holds if and only if $\langle A'_1, A'_1\rangle + \langle A'_2, A'_2\rangle + \langle A'_3, A'_3\rangle = 0$. Moreover, $[\alpha, 0]$ is indecomposable if and only if $\mathfrak{a} = \mathfrak{a}_+ = \mathrm{span}\{A'_1, A'_2, A'_3\}$. For admissibility we have to check only (A_0) and (B_0). For an indecomposable cohomology class $[\alpha, 0]$ these conditions are satisfied if and only if $\alpha \ne 0$. In particular, if $\mathcal{H}^2_Q(\mathbb{H}, \Phi_{\mathbb{H}}, \mathfrak{a})_0$ is not empty, then $\mathfrak{a} = \mathfrak{a}_+$ is isomorphic to $\mathbb{R}^{1,1}$, $\mathbb{R}^{1,2}$ or $\mathbb{R}^{2,1}$. Now we study the action of $G_{\mathbb{H},\Phi_{\mathbb{H}},\mathfrak{a}}$ on $\mathcal{H}^2_Q(\mathbb{H}, \Phi_{\mathbb{H}}, \mathfrak{a})_0$. We have $\mathrm{Aut}(\mathbb{H}, \Phi_{\mathbb{H}}) \cong \mathbb{H}^*$, where an element $q \in \mathbb{H}^*$ acts by right multiplication (denoted by R_q) on $\mathbb{H}$. Take $q \in \mathbb{H}^*$. Then $q = rq_0$ with $r \in \mathbb{R}$, $r > 0$ and $q_0 \in Sp(1)$. Take α as in (11) and set

$$R_q^*\alpha(1,i) = \bar{A}_1, \qquad R_q^*\alpha(1,j) = \bar{A}_2, \qquad R_q^*\alpha(1,k) = \bar{A}_3.$$

Let $\lambda : Sp(1) \to SO(3)$ be the double covering. Then we have

$$(\bar{A}_1, \bar{A}_2, \bar{A}_3) = r^2(A'_1, A'_2, A'_3) \cdot \lambda(\bar{q}_0).$$

Hence, replacing α by an element in the same $G_{\mathbb{H},\Phi_{\mathbb{H}},\mathfrak{a}}$-orbit we may assume that A'_1, A'_2, A'_3 are pairwise orthogonal and that

$$\langle A'_1, A'_1\rangle = -1, \qquad \langle A'_2, A'_2\rangle = 1, \qquad A'_3 = 0$$

or

$$\langle A'_1, A'_1\rangle = -1, \qquad \langle A'_2, A'_2\rangle = t, \qquad \langle A'_3, A'_3\rangle = 1 - t, \quad 0 < t \le 1/2$$

or

$$\langle A'_1, A'_1\rangle = -t, \qquad \langle A'_2, A'_2\rangle = t - 1, \qquad \langle A'_3, A'_3\rangle = 1, \quad 0 < t \le 1/2,$$

which immediately implies the assertion. □

Now let us consider $(\mathfrak{l}_0, \Phi_{\mathfrak{l}_0})$. Let $\mathfrak{a}_0$ and $[\alpha_0, \gamma_0] \in \mathcal{H}^2_Q(\mathfrak{l}_0, \Phi_{\mathfrak{l}_0}, \mathfrak{a}_0)_0$ be as defined in Section 5.

Proposition 7.3. *The orbit space* $\mathcal{H}^2_Q(\mathfrak{l}_0, \Phi_{\mathfrak{l}_0}, \mathfrak{a}_0)_0/G_{\mathfrak{l}_0,\Phi_{\mathfrak{l}_0},\Phi_{\mathfrak{a}_0}}$ *contains exactly one element (represented by* $[\alpha_0, \gamma_0]$*). If* $\mathfrak{a}$ *is a vector space with orthogonal quaternionic grading which is not isomorphic to* $\mathfrak{a}_0$ *or to* $\mathfrak{a}_0^- := (\mathfrak{a}_0, -\langle\cdot\,,\cdot\rangle_0, \Phi_{\mathfrak{a}_0})$*, then* $\mathcal{H}^2_Q(\mathfrak{l}_0, \Phi_{\mathfrak{l}_0}, \mathfrak{a})_0$ *is empty.*

Proof. Let $\mathfrak{a}$ be a vector space with orthogonal quaternionic grading. It is easy to verify that for

$$Z^2_{\mathfrak{l}_0} := \left\{\alpha \in C^2(\mathfrak{l}_0, \mathfrak{a})^{Sp(1)} \;\middle|\; \begin{array}{l} i\alpha(1, I) + j\alpha(1, J) + k\alpha(1, K) = 0 \\ \alpha((\mathfrak{l}_0)_+, (\mathfrak{l}_0)_+) = \alpha((\mathfrak{l}_0)_-, (\mathfrak{l}_0)_-) = 0 \end{array}\right\}$$

the map

$$Z^2_{\mathfrak{l}_0} \longrightarrow H^2(\mathfrak{l}_0, \mathfrak{a})^{Sp(1)}, \qquad \alpha \longmapsto [\alpha]$$

is correctly defined (similar computation as in the proof of Lemma 5.1) and an isomorphism. Now assume that $\mathcal{H}^2_Q(\mathfrak{l}_0, \Phi_{\mathfrak{l}_0}, \mathfrak{a})_0$ is not empty and take $[\alpha, \gamma] \in \mathcal{H}^2_Q(\mathfrak{l}_0, \Phi_{\mathfrak{l}_0}, \mathfrak{a})_0$. We may assume that $\alpha \in Z^2_{\mathfrak{l}_0}$. We set

$$A_1 := -\alpha(i, I), \qquad A_2 := -\alpha(j, J), \qquad A_3 = -\alpha(k, K)$$

and $\gamma(I, J, K) = 2c$. Since (α, γ) is indecomposable we have

$$\mathfrak{a} = \mathfrak{a}_- = \mathbb{H} \otimes_{\mathbb{R}} \operatorname{span}\{A_1, A_2, A_3\}.$$

By the same computation as in the proof of Lemma 5.1 the equation

$$\frac{1}{2}\langle\alpha \wedge \alpha\rangle(1, q, P, Q) = \mathrm{d}\gamma(1, q, P, Q)$$

for all $q \in \mathbb{H}$, $P, Q \in \{I, J, K\}$ yields

$$\langle q_1 A_1, q_2 A_2\rangle_{\mathfrak{a}} = \langle q_1 A_1, q_2 A_3\rangle_{\mathfrak{a}} = \langle q_1 A_2, q_2 A_3\rangle_{\mathfrak{a}} = -\frac{1}{2}\langle q_1, q_2\rangle_{\mathbb{H}} \cdot \gamma(I, J, K) = -c\langle q_1, q_2\rangle_{\mathbb{H}}.$$

This implies

$$0 = \langle p(A_1 + A_2 + A_3), qA_i\rangle = \langle pA_i, qA_i\rangle_{\mathfrak{a}} - 2c\langle p, q\rangle_{\mathbb{H}},$$

thus $\langle pA_i, qA_i\rangle_{\mathfrak{a}} = 2c\langle p, q\rangle_{\mathbb{H}}$ for $i = 1, 2, 3$, $p, q \in \mathbb{H}$. Assume $c = 0$. Then $\alpha = 0$ and $\gamma((\mathfrak{l}_0)_+, (\mathfrak{l}_0)_+, \mathfrak{l}_0) = 0$. Then it follows in the same way as in the proof of Proposition 7.1, that $[\alpha, \gamma]$ is not admissible. Hence $c \neq 0$. If $c > 0$, then by the above equations $\mathfrak{a}$ is isomorphic to $\mathfrak{a}_0$ as a vector space with orthogonal quaternionic grading and if $c < 0$, then $\mathfrak{a}$ is isomorphic to $\mathfrak{a}_0^-$.

Hence we may assume that $\mathfrak{a} = \mathfrak{a}_0$, $\alpha = |c|^{1/2}\alpha_0$, or $\mathfrak{a} = \mathfrak{a}_0^-$, $\alpha = |c|^{1/2}\alpha_0$. Let us now show that $[\alpha, \gamma] = [\alpha, \tilde{\gamma}] \in \mathcal{H}^2_Q(\mathfrak{l}, \Phi_{\mathfrak{l}}, \mathfrak{a})_0$ with $\tilde{\gamma}(I, J, K) = 2c$ and $\tilde{\gamma}((\mathfrak{l}_0)_-, (\mathfrak{l}_0), (\mathfrak{l}_0)) = 0$. We consider the subspaces

$$\{\langle\alpha \wedge \tau\rangle \in C^3(\mathfrak{l}_0)^{Sp(1)} \mid \tau \in Z^1(\mathfrak{l}_0, \mathfrak{a})^{Sp(1)}\} \subset \{\gamma' \in Z^3(\mathfrak{l}_0)^{Sp(1)} \mid \gamma'(I, J, K) = 0\}$$

of $C^3(\mathfrak{l}_0)^{Sp(1)}$. By definition of $\mathcal{H}^2_Q(\mathfrak{l}, \Phi_{\mathfrak{l}}, \mathfrak{a})$ it suffices to show that both subspaces are equal. Note that the latter of these spaces is 8-dimensional. Since $Z^1(\mathfrak{l}_0, \mathfrak{a})^{Sp(1)}$ is also 8-dimensional it remains to prove that the map

$$Z^1(\mathfrak{l}_0, \mathfrak{a})^{Sp(1)} \ni \tau \longmapsto \langle\alpha \wedge \tau\rangle \in C^3(\mathfrak{l}_0)^{Sp(1)}$$

is injective. Assume that τ is in the kernel of this map. Then we have

$$0 = \langle\alpha \wedge \tau\rangle(1, q, Q) = \langle\alpha(q, Q), \tau(1)\rangle - \langle\alpha(1, Q), \tau(q)\rangle = 2\langle\alpha(q, Q), \tau(1)\rangle$$

for all $q \in \operatorname{Im}\mathbb{H}$ and all $Q \in \{I, J, K\}$. Since

$$\mathfrak{a} = \operatorname{span}\{\alpha(q, Q) \mid q \in \operatorname{Im}\mathbb{H}, Q \in \{I, J, K\}\}$$

this implies $\tau(1) = 0$, hence $\tau = 0$.

Hence we may assume that $\mathfrak{a} = \mathfrak{a}_0$, $\alpha = |c|^{1/2}\alpha_0$, and $\gamma = |c|\gamma_0$ or $\mathfrak{a} = \mathfrak{a}_0^-$, $\alpha = |c|^{1/2}\alpha_0$, and $\gamma = -|c|\gamma_0$. If we now take $S = |c|^{-1/3}\mathrm{Id} \oplus |c|^{-1/6}\mathrm{Id} : (\mathfrak{l}_0)_+ \oplus (\mathfrak{l}_0)_- \to (\mathfrak{l}_0)_+ \oplus (\mathfrak{l}_0)_-$, then $S \in \operatorname{Aut}(\mathfrak{l}_0, \Phi_{\mathfrak{l}_0})$ and we get $(S^*\alpha, S^*\gamma) = (\alpha_0, \pm\gamma_0)$. □

As a consequence of Theorem 6.1, Eq. (5) and Propositions 7.1–7.3 we obtain the following classification.

Theorem 7.4. *If* $(\mathfrak{g}, \Phi_{\mathfrak{g}}, \langle\cdot\,,\cdot\rangle_{\mathfrak{g}})$ *is a hyper-Kähler symmetric triple which is associated with an indecomposable hyper-Kähler symmetric space of index 4, then it is isomorphic to* $(\mathbb{H}, \Phi_{\mathbb{H}}, -\langle\cdot\,,\cdot\rangle_{\mathbb{H}})$ *or to* $\mathfrak{d}_{\alpha,\gamma}(\mathfrak{l}, \Phi_{\mathfrak{l}}, \mathfrak{a})$ *for exactly one of the data in the following table:*

$(\mathfrak{l}, \Phi_{\mathfrak{l}})$	$\mathfrak{a}$	α	γ	*Parameters*
$(\mathbb{H}, \Phi_{\mathbb{H}})$	$\mathfrak{a} = \mathfrak{a}_+ = \mathbb{R}^{1,1}$	α'	0	–
	$\mathfrak{a} = \mathfrak{a}_+ = \mathbb{R}^{1,2}$	α_r	0	$0 < r \le \pi/4$
	$\mathfrak{a} = \mathfrak{a}_+ = \mathbb{R}^{2,1}$	α_s	0	$0 < s \le \pi/4$
$(\mathfrak{l}_0, \Phi_{\mathfrak{l}_0})$	$\mathfrak{a}_0$	α_0	γ_0	–

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